

AN UPDATED SURVEY ON SOME RESULTS ON ENERGY OF GRAPHS

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ABSTRACT

The eigenvalue of a graph G is the eigenvalue of its adjacency matrix. The energy $E(G)$ of G is the sum of absolute values of the eigenvalues. A natural question arises: How the energy of a given graph G can be related with the graph obtained from G by means of some graph operations? In order to answer this question, we have considered two graphs namely, splitting graph $S(G)$ and shadow graph $D_2(G)$. It has been proven that $E(S'(G)) = \sqrt{5}E(G)$ and $E(D_2(G)) = 2E(G)$.

1. INTRODUCTION

All graphs considered here are simple, finite and undirected. For standard terminology and notations related to graph theory, we follow Balakrishnan and Ranganathan [2] while for algebra we follow Lang [7].

The Adjacency matrix $A(G)$ of a graph G with vertices $v_1, v_2, \dots, v_n$ is an $n \times n$ matrix $[a_{ij}]$ such that $a_{ij} = 1$ if v_i is adjacent to v_j , and 0 otherwise.

The eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ of the graph G are the eigenvalues of its adjacency matrix. The set of eigenvalues of the graph with their multiplicities is known as spectrum of the graph.

In 1978 Gutman [5] defined the energy of a Graph G as the sum of absolute values of the eigenvalues of graph G and denoted it by $E(G)$, Hence,

$$E(G) = \sum_{i=1}^n |\lambda_i|$$

In 2004, Bapat and Pati [3] proved that if the energy of a graph is rational then it must be an even integer, while Pirzada and Gutman [9] established that the energy of a graph is never the square root of an odd integer. A brief account of graph energy is given in [1] as well as in the books [4,8]. Some fundamental results on graph energy are also reported in the thesis of Sriraj [10].

The present work is intended to relate the graph energy to larger graphs obtained from the given by means of some graph operations.

Let $A \in R^{m \times n}$, $B \in R^{p \times q}$, Then the Kronecker product (or tensor product) of A and B is defined as the matrix

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}$$

Proposition 1.1 : [6] Let $A \in M^m$ and $B \in M^n$. Furthermore, let λ be an eigenvalue of matrix A with corresponding eigen vector x , and μ an eigen value of matrix B with

corresponding eigenvector y . Then λ_μ is an eigen value of $A \otimes B$ with corresponding eigenvector $x \otimes y$.

2. ENERGY OF SPLITTING GRAPH

The splitting graph, $S^1(G)$ of a graph G is obtained by adding to each vertex v a new vertex v' , such that v' is adjacent to every vertex that is adjacent to v in G . We prove the following result.

Theorem 2.1. $E(S^1(G)) = \sqrt{5}E(G)$.

Proof: Let $v_1, v_2, \dots, v_n$ be the vertices of the graph G . Then its adjacency matrix is given by

		v_1	v_2	v_3	...	v_n
	v_1	0	a_{12}	a_{13}	...	a_{1n}
	v_2	a_{21}	0	a_{23}	...	a_{2n}
$A(G) =$	v_3	a_{31}	a_{32}	0	...	a_{3n}
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	
	v_n	a_{n1}	a_{n2}	a_{n3}	...	0

Let $u_1, u_2, \dots, u_n$ be the vertices corresponding to $v_1, v_2, \dots, v_n$ which are added in G to obtain $S^1(G)$, such that, $N(v_i) = N(u_i), i = 1, 2, \dots, n$. Then $A(S^1(G))$ can be written as a block matrix as follows

		v_1	v_2	v_3	...	v_n	u_1	u_2	u_3	...	u_n
	v_1	0	a_{12}	a_{13}	...	a_{1n}	0	a_{12}	a_{13}	...	a_{1n}
	v_2	a_{21}	0	a_{23}	...	a_{2n}	a_{21}	0	a_{23}	...	a_{2n}
	v_3	a_{31}	a_{32}	0	...	a_{3n}	a_{31}	a_{32}	0	...	a_{3n}
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$A(S^1(G))$	v_n	a_{1n}	a_{2n}	a_{3n}	...	0	a_{1n}	a_{2n}	a_{3n}	...	0
	u_1	0	a_{12}	a_{13}	...	a_{1n}	0	0	0	...	0
	u_2	a_{21}	0	a_{23}	...	a_{2n}	0	0	0	...	0
	u_3	a_{31}	a_{32}	0	...	a_{3n}	0	0	0	...	0
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
	u_n	a_{1n}	a_{2n}	a_{3n}	...		0	0	0	...	0

That is

$$A(S^1(G)) = \begin{bmatrix} A(G) & A(G) \\ A(G) & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \otimes A(G)$$

Hence,

$$\text{spec}(S^1(G)) = \left(\left(\frac{1+\sqrt{5}}{2} \right) \lambda_i \quad \left(\frac{1-\sqrt{5}}{2} \right) \lambda_i \right)_n$$

Where $\lambda_i, i = 1, 2, \dots, n$, are the eigen values of G , while $\left(\frac{1 \pm \sqrt{5}}{2} \right)$ are the eigenvalues of $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. Here,

$$\begin{aligned} E(S^1(G)) &= \sum_{i=1}^n \left| \left(\frac{1 \pm \sqrt{5}}{2} \right) \lambda_i \right| = \sum_{i=1}^n |\lambda_i| \left[\frac{\sqrt{5}+1}{2} + \frac{\sqrt{5}-1}{2} \right] \\ &= \sqrt{5} \sum_{i=1}^n |\lambda_i| = \sqrt{5} E(G). \end{aligned}$$

Illustration 2.2 : Consider the cycle C_4 and its spitting graph $S^1(C_4)$. It is obvious that $E(C_4) = 4$ as $\text{spec}(C_4) = \begin{pmatrix} -2 & 2 & 0 \\ 1 & 1 & 2 \end{pmatrix}$.

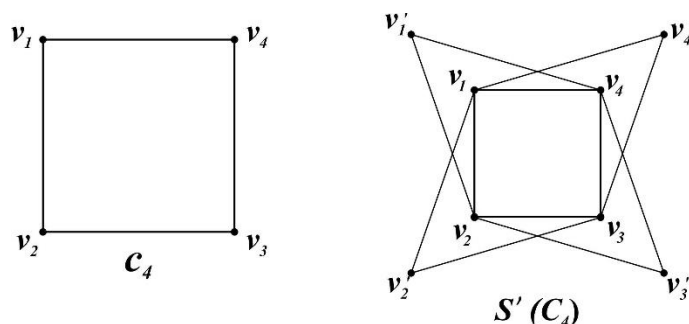


Figure 1

		v_1	v_2	v_3	v_4	v'_1	v'_2	v'_3	v'_4
	v_1	0	1	0	1	0	1	0	1
	v_2	1	0	1	0	1	0	1	0
	v_3	0	1	0	1	0	1	0	1
$A(S'(C_4)) =$	v_4	1	0	1	0	1	0	1	0
	v'_1	0	1	0	1	0	1	0	1
	v'_2	1	0	1	0	1	0	1	0
	v'_3	0	1	0	1	0	1	0	1
	v'_4	1	0	1	0	1	0	1	0

Therefore, $\text{spec}(S'(C_4)) = \begin{pmatrix} 1+\sqrt{5} & 1-\sqrt{5} & -1-\sqrt{5} & -1+\sqrt{5} & 0 \\ 1 & 1 & 1 & 1 & 4 \end{pmatrix}$

Hence, $E(S'(C_4)) = 4\sqrt{5} = \sqrt{5} E(C_4)$

3. ENERGY OF SHADOW GRAPH

The shadow graph $D_2(G)$ of a connected graph G is constructed by taking two copies of G , say G' and G'' . Join each vertex u' in G' to the neighbours of the corresponding vertex u' in G'' .

Theorem 3.1 $E(D_2(G)) = 2E(G)$

Proof : Let $v_1 v_2 \dots v_n$ be the vertices of the Graph G . Then its adjacency matrix is same as in the proof Theorem 2.1. Consider the second copy of graph G with vertices $v_1, v_2, v_3 \dots u_n$ to obtain $D_2(G)$, such that $N(v_i) = N(u_i)$, $i = 1, 2, \dots, n$. Then $A(D_2(G))$ can be written as a block matrix as follows :

		v_1	v_2	v_3	...	v_n	u_1	u_2	u_3	...	u_n
	v_1	0	a_{12}	a_{13}	...	a_{1n}	0	a_{12}	a_{13}	...	a_{1n}
	v_2	a_{21}	0	a_{23}	...	a_{2n}	a_{21}	0	a_{23}	...	a_{2n}
	v_3	a_{31}	a_{32}	0	...	a_{3n}	a_{31}	a_{32}	0	...	a_{3n}
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$A(D_2(G))$ =	v_n	a_{1n}	a_{2n}	a_{3n}	...	0	a_{1n}	a_{2n}	a_{3n}	...	0
	u_1	0	a_{12}	a_{13}	...	a_{1n}	0	0	0	...	0
	u_2	a_{21}	0	a_{23}	...	a_{2n}	0	0	0	...	0
	u_3	a_{31}	a_{32}	0	...	a_{3n}	0	0	0	...	0
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
	u_n	a_{1n}	a_{2n}	a_{3n}	...		0	0	0	...	0

That is,

$$A(D_2(G)) = \begin{bmatrix} A(G) & A(G) \\ A(G) & A(G) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \otimes A(G)$$

Hence,

$\text{spec}((D_2(G))) = \begin{pmatrix} 0 & 2\lambda_i \\ n & n \end{pmatrix}$ where λ_i , $i = 1, 2, \dots, n$, are the eigenvalues of G , while 0, 2 are eigenvalues of $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

Here,

$$E(D_2(G)) = \sum_{i=1}^n |2\lambda_i| = 2 \sum_{i=1}^n |\lambda_i| = 2E(G)$$

Illustration 3.2. : Consider the cycle C_4 and its shadow graph $D_2(C_4)$. From the previous example if its known that $E(C_4) = 4$ and $\text{spec}(C_4) = \begin{pmatrix} -2 & 2 & 0 \\ 1 & 1 & 2 \end{pmatrix}$

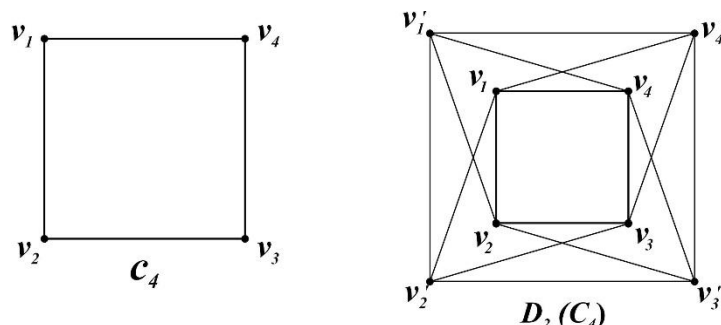


Figure-2

		v_1	v_2	v_3	v_4	v'_1	v'_2	v'_3	v'_4
	v_1	0	1	0	1	0	1	0	1
	v_2	1	0	1	0	1	0	1	0
	v_3	0	1	0	1	0	1	0	1
$A(D_2(C_4)) =$	v_4	1	0	1	0	1	0	1	0
	v'_1	0	1	0	1	0	1	0	1
	v'_2	1	0	1	0	1	0	1	0
	v'_3	0	1	0	1	0	1	0	1
	v'_4	1	0	1	0	1	0	1	0

Therefore, $\text{spec}(D_2(C_4)) = \begin{pmatrix} 4 & -4 & 0 \\ 1 & 1 & 6 \end{pmatrix}$. Hence,

$$E(D_2(C_4)) = 8 = 2E(C_4).$$

4. CONCLUDING REMARKS

The energy a graph is one of the emerging concepts within graph theory. This concept serve as a frontier between chemistry and mathematics. The energy of many graphs is known. But we have take up the problem to investigate the energy of a graph obtained by means of some graph operations on a given graph and it has been revealed that two energy of the new graph is a multiple of energy of a given graph.

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